数理解析研究所講究録
395巻 1980年 34-46

ON THE FAITHFUL PRIME RADICAL

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1. Introduction. Let R be a ring with identity. The faithful prime left R-modules in the sense of Page [7], together with the zero module, form a torsionfree class of R-mod. The corresponding torsion class is $\{_R A \mid A$ has no faithful prime images$\}$ In [2] Handelman and Lawrence has posed the following question: What can be said about this torsion theory ? The purpose of this note is to investigate this problem.

After some preliminaries, we shall define, in the section 3, a preradical s of R-mod and discuss its basic properties. In the section 4 we shall treat the case where the left linear topology associated with s has the smallest element. In the final section we shall discuss the problem using the localization with respect to the faithful prime radical β_0, defined in Nicholson and Watters [6].

Throughout this note R will denote an associative ring with identity and all R-modules will be unitary left R-modules. The category of unitary left R-modules is denoted by R-mod. For the notions and terminologies about torsion theories we refer to Stenström [8].

2. Preliminaries. A functor t: R-mod $\to$ R-mod is called a preradical if $t(M)$ is a submodule of M and $f(t(M)) \leq t(N)$ for each morphism $f: M \to N$ in R-mod. A preradical t is called a radical if $t(M/t(M)) = 0$ for all $M \in$ R-mod. It is called left exact if $t(N) = N \cap t(M)$ whenever $N \leq M$ in R-mod. To each preradical t we can associate two subclasses of R-mod, namely

$$T(t) = \{{}_R A \mid t(A) = A\} \text{ and } F(t) = \{{}_R B \mid t(B) = 0\}.$$

Page [7] has defined a prime R-module as a nonzero R-module whose left annihilator in R is the same as that of any of its nonzero submodules, i.e. an R-module M is prime if and only if $M \neq 0$ and $\ell_R(M) = \ell_R(Rx)$ for every nonzero element x in M. Then M is faithful prime if and only if $M \neq 0$ and $\ell_R(M') = 0$ for every nonzero submodule M' of M, or equivalently, $M \neq 0$ and $\ell_R(Rx) = 0$ for every nonzero element x in M.

From this definition we have at once

Proposition 2.1. The following conditions for a ring R are equivalent:

(1) R is a prime ring.

(2) ${}_R R$ is faithful prime.

(3) There exists a faithful prime R-module.

As is easily seen, the faithful prime R-modules, together with the zero module, form a torsionfree class of R-mod and the corresponding torsion class is exactly

$$\{{}_R A \mid A \text{ has no faithful prime images}\}.$$

For each R-module M, following Nicholson and Watters [6], define the faithful prime radical $\beta_0(M)$ of M as

$$\beta_0(M) = \bigcap \{M' \leq M \mid M/M' \text{ is faithful prime}\}.$$

The functor β_0: R-mod $\to$ R-mod is then a left exact radical,

$T(\beta_0) = \{_R A \mid A$ has no faithful prime images$\}$ and $F(\beta_0) = \{_R B \mid$ B is faithful prime$\} \cup \{0\}$. Hence the problem posed by Handelman and Lawrence [2] means investigating the torsion theory $(T(\beta_0)$, $F(\beta_0))$.

As to β_0 we have

Proposition 2.2. The following conditions are equivalent:

(1) R is a prime ring.

(2) $\beta_0(R) = 0$.

(3) $\beta_0 \neq 1$.

Furthermore we have

Proposition 2.3. The following conditions are equivalent:

(1) $\beta_0 = 0$.

(2) R is simple (R has no non-trivial two-sided ideals.)

Proof. (1) $\Rightarrow$ (2). Let $\underline{a}$ be a two-sided ideal $\neq R$. Then $R/\underline{a}$ is faithful prime and so $\ell_R(R(1 + \underline{a})) = 0$. Since $bR(1 + \underline{a}) = 0$ for every b in $\underline{a}$, we see that $\underline{a} = 0$. (2) $\Rightarrow$ (1). Let $M \neq 0$ be an R-module. Then, for each $x \neq 0$ in M, $\ell_R(Rx)$ is a two-sided ideal $\neq R$ and thus we have $\ell_R(Rx) = 0$.

3. The Preradical s. From now on, by means of Propositions 2.2 and 2.3, we can assume that R is prime and is not simple.

For each R-module M, define a functor s: R-mod $\to$ R-mod as

$s(M) = \{x \in M \mid Ix = 0$ for some two-sided ideal $I \neq 0\}$.

Then, since R is prime, s(M) is a submodule of M and s is a left exact preradical. We can also describe s(M) as

$$s(M) = \{x \in M \mid aRx = 0 \text{ for some } a \neq 0 \text{ in } R\}$$

and hence $s(M) = 0$ if and only if $M = 0$ or M is faithful prime. Therefore the smallest radical $\bar{s}$ larger than s coincides with β_0

and $s(M)$ is essential in $\beta_0(M)$ for every R-module M.

Nonzero two-sided ideals in a prime ring are essential in R as R-modules and so we have

Proposition 3.1. $s(M) \leqq Z(M)$ for every R-module M.

It follows from this $s \leqq Z$ and hence $\beta_0 = \bar{s} \leqq \bar{Z} = G$. As was pointed out by [6, p. 240] in general β_0 does not coincide with G.

Theorem 3.2. The following conditions are equivalent:

(1) $\beta_0 = G$.

(3) $\beta_0(M)$ is essential in $G(M)$ for every R-module M.

(3) $s(M)$ is essential in $G(M)$ for every R-module M.

(4) $F(s) = F(Z)$.

Proof. We only show that (3) $\Rightarrow$ (4). By Proposition 3.1 $F(s) \geqq F(Z)$. Assume that there exists an R-module M such that $s(M) = 0$ and $Z(M) \neq 0$. Then $Z(M)$ is a nonzero submodule of $G(M)$ and so $0 \neq s(M) \cap Z(M) = s(M) = 0$, a contradiction.

An R-module $M = 0$ is strongly prime, following Handelman and Lawrence [2], if for each $x \neq 0$ in M there exists a finite subset F in R such that $\ell_R(Fx) = 0$. We call a ring R strongly prime if ${}_RR$ is strongly prime. Then by [2, Corollary 3 to Proposition V.1] if R is strongly prime, then an R-module $M \neq 0$ is strongly prime if and only if $Z(M) = 0$.

On the other hand, Nicholson and Watters [6] has defined the strongly prime radical $\beta(M)$ of an R-module M as

$$\beta(M) = \bigcap \{M' \leqq M \mid M/M' \text{ is strongly prime}\}.$$

Hence if R is strongly prime, then $\beta(M) = \bar{Z}(M) = G(M)$ for every R-module M. However R is non-singular and so we have $Z = G = \beta$. Thus we have $\beta_0 \leqq \beta$.

As an application of Theorem 3.2, we have

Corollary 3.3. For a strongly prime ring R the following conditions are equivalent:

(1) $\beta_0 = \beta$.

(2) $\beta_0(M)$ is essential in $Z(M)$ for every R-module M.

(3) $s(M)$ is essential in $Z(M)$ for every R-module M.

(4) $F(s) = F(Z)$.

We now determine the left linear topology $L(s)$ associated with s. For a left ideal $\underline{a}$ in R, $s(R/\underline{a}) = R/\underline{a}$ if and only if $1 + \underline{a} \in s(R/\underline{a})$, i.e. if and only if $I \leqq \underline{a}$ for some two-sided ideal $I \neq 0$ in R. So $L(s)$ is a bounded left linear topology with a basis consisting of all the nonzero two-sided ideals in R. $L(s)$ can be also described as

$$\begin{aligned} L(s) &= \{\underline{a} \leqq {}_R R \mid \underline{b} \leqq \underline{a} \text{ for some right ideal } b \neq 0\} \\ &= \{\underline{a} \leqq {}_R R \mid cR \leqq \underline{a} \text{ for some } c \neq 0 \text{ in } R\}. \end{aligned}$$

In case R is commutative, or more generally R is left duo, then we have

$$L(s) = \{\underline{a} \leqq {}_R R \mid \underline{a} \neq 0\} \text{ and}$$

$$s(M) = \{x \in M \mid \ell_R(x) \neq 0\}$$

for each R-module M. Since R is prime, $\ell_R(x) \neq 0$ means $\ell_R(x)$ essential in R and hence we have $s(M) = Z(M)$ for every R-modue M. Furthermore $Z(R) = s(R) = 0$. Therefore $s = Z$ is a radical.

Theorem 3.4. If R is left duo, then $\beta_0(M) = Z(M)$ for every R-module M.

Turning to a general case, s does not coincide with Z, as was pointed out by [2, p. 222]. However, $s = Z$ if and only if essential left ideals in R contain nonzero two-sided ideals.

Besides commutative rings and left duo rings, examples of rings with this property are (prime) rings with nonzero socles, and left fully bounded (prime) rings.

In general, as is well-known, every left Gabriel topology on R is closed under finite products. However we have

<u>Proposition</u> 3.5. L(s) is closed under finite products.

Since s is a left exact preradical, it is a radical if and only if $\underline{a}$ is a left ideal and there exists a left ideal $\underline{b} \in L(s)$ such that $(\underline{a}:c) \in L(s)$ for every $c \in \underline{b}$, then $\underline{a} \in L(s)$. It is easy to see that this condition is satisfied if R is either left Noetherian or subdirectly irreducible.

4. 3-<u>fold</u> <u>Torsion</u> <u>Theories</u>. As was pointed out just above, if R is subdirectly irreducible then s becomes a radical. Moreover we have

<u>Proposition</u> 4.1. The following conditions are equivalent:

(1) R is subdirectly irreducible.

(2) L(s) has the smallest element.

Proof. L(s) has the smallest element if and only if $\bigcap\{\underline{a} \mid \underline{a} \in L(s)\} \in L(s)$ and this is so if and only if $\bigcap\{I \mid I \text{ is a two-sided ideal} \neq 0\} \neq 0$, i.e. R is subdirectly irreducible.

As to a left linear topology with the smallest element, we have

<u>Proposition</u> 4.2. Let I be a left ideal in R. Then

(1) $L = \{\underline{a} \leqq {}_R R \mid I \leqq \underline{a}\}$ is a filter.

(2) L is a left linear topology on R if and only if I is a two-sided ideal in R.

(3) L is a left Gabriel topology on R if and only if I is an idempotent two-sided ideal in R.

<u>Proposition</u> 4.3. For a left exact preradical t, the following conditions are equivalent:

(1) $T(t)$ is closed under direct products.

(2) $L(t)$ has the smallest element I.

(3) There exists a two-sided ideal J in R such that $t(M) = r_M(J)$ for every R-module M.

(4) There exists a two-sided ideal K in R such that $T(t) = \{_R A \mid KA = 0\}$.

Furthermore, if this is the case, $I = J = K$ and t is a radical if and only if I is idempotent.

Now we assume that R is subdirectly irreducible. Then s is a radical and so $s = \beta_0$. $L(\beta_0)$ has the smallest element $I_0 = \bigcap \{I \mid I \text{ is a two-sided ideal} \neq 0\}$ which is nonzero idempotent by Proposition 4.2. $T(\beta_0)$ is closed under direct products by Proposition 4.3 and hence we can find a subclass C of R-mod such that $(C, T(\beta_0), F(\beta_0))$ forms a 3-fold torsion theory in the sense of [3]. Furthermore using Proposition 4.3 again, $\beta_0(M) = r_M(I_0)$ for every R-module M and we have

$$\begin{aligned} T(\beta_0) &= \{_R A \mid I_0 A = 0\} \\ &= \{_R A \mid IA = 0 \text{ for some two-sided ideal } I \neq 0\} \end{aligned}$$

and

$$\begin{aligned} F(\beta_0) &= \{_R B \mid r_B(I_0) = 0\} \\ &= \{_R B \mid r_B(I) = 0 \text{ for all two-sided ideals } I \neq 0\}. \end{aligned}$$

By [3, Lemma 2.1],

$$\begin{aligned} C &= \{_R M \mid I_0 M = M\} \\ &= \{_R M \mid IM = M \text{ for all two-sided ideals } I \neq 0\}. \end{aligned}$$

Summarizing these facts, we have

<u>Theorem</u> 4.4. The following conditions are equivalent:

(1) R is subdirectly irreducible.

(2) There exists a subclass C of R-mod such that $(C, T(\beta_0), F(\beta_0))$ forms a 3-fold torsion theory.

(3) There exists an idempotent two-sided ideal I_0 such that

$$\beta_0(M) = r_M(I_0)$$

for every R-module M.

Moreover, if this is the case, C can be given by

$$\{{}_RM \mid IM = M \text{ for all two-sided ideals } I \neq 0\}.$$

Proof. We only show that (3) $\Rightarrow$ (1). By Proposition 4.3, $L(\beta_0)$ has the smallest element I_0. Let I be any two-sided ideal $\neq 0$ in R. Then $I \in L(s) \leqq L(\beta_0)$. So $I_0 \leqq I$ and $I_0 \leqq \bigcap\{I \mid I$ is a two-sided ideal $\neq 0\}$. If $I_0 = 0$, then $\beta_0(M) = r_M(I_0) = M$ for every R-module M. Hence we have $\beta_0 = 1$ which contradicts to the assumption that R is prime by Proposition 2.2. Thus we have $\bigcap\{I \mid I \text{ is a two-sided ideal } \neq 0\} \neq 0$.

Assume again that R is subdirectly irreducible. Then $(C, T(\beta_0), F(\beta_0))$ forms a 3-fold torsion theory for some subclass C of R-mod. Can this 3-fold torsion theory be extended to a 4-fold torsion theory ? If this can be extended to a right-hand side and becomes a 4-fold torsion theory, then $F(\beta_0)$ becomes a torsion class and we have $F(\beta_0)$ = R-mod because $R \in F(\beta_0)$. Hence $\beta_0 = 0$ which contradicts to the assumption that R is not simple by Proposition 2.3. On the other hand, if this can be extended to a left-hand side, then this must have length 2 by [3, Proposition 4.6] and so $C = F(\beta_0)$, a contradiction.

Thus we have

Proposition 4.5. If R is subdirectly irreducible, then the 3-fold torsion theory $(C, T(\beta_0), F(\beta_0))$ can not be extended to any 4-fold torsion theory and hence it has length 3.

5. Localizations. Let M be an R-module and t a left exact radical. Let M_t be the localization of M with respect to t and $\eta_M: M \to M_t$ the canonical mapping. As is well-known, both $\mathrm{Ker}(\eta_M)$ and $\mathrm{Coker}(\eta_M)$ are in $T(t)$, M_t is in $F(t)$ and M_t is t-injective. The following lemma shows that these properties characterize the localization M_t.

Lemma 5.1 ([4, 2.2]). Let M and X are R-modules and let $f: M \to X$ be an R-homomorphism. Suppose that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are in $T(t)$. Then:

(1) There exists a unique R-homomorphism $h: X \to M_t$ making the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\eta_M"'] & X \arrow[dl, "h"] \\ M_t & \end{tikzcd}$$

commutative.

(2) $\mathrm{Ker}(h) = t(X)$.

(3) h is an epimorphism if and only if $X/t(X)$ is t-injective.

Note that, if in particular M and X are rings and f is a ring homomorphism, then h is a ring homomorphism.

Let M be again an R-module and t a left exact radical. For the natural homomorphism $\pi: M \to M/t(M)$, by Lemma 5.1, there exists a (unique) R-monomorphism h such that the diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\eta_M"'] & M/t(M) \arrow[dl, "h"] \\ M_t & \end{tikzcd}$$

is commutative. Again by Lemma 5.1, $M/t(M) \simeq M_t$ if and only if $M/t(M)$ is t-injective. Thus we have

Proposition 5.2 (cf. [1, Théorème 1]). The following conditions are equivalent for a left exact radical t:

(1) $M/t(M) \simeq M_t$ for every R-module M.

(2) Each element of F(t) is t-injective.

(3) F(t) is closed under homomorphic images.

In case $t(R) = 0$, these are also equivalent to the following conditions.

(4) $t = 0$.

(5) Every R-module is t-injective.

(6) $M \simeq M_t$ for every R-module M.

Since β_0 is a left exact radical and $\beta_0(R) = 0$, we can apply Proposition 5.2 to β_0.

Let M be an R-module and t a left exact radical. Consider the following diagram:

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\eta_M"'] & M/t(M) \arrow[r, "\nu"] & E(M/t(M)) \\ M_t & & \end{tikzcd}$$

where ν denotes the inclusion map. If we assume that

(*) $$_RN \in F(t) \Rightarrow E(N)/N \in T(t),$$

then by Lemma 5.1 there exists an R-isomorphism $h\colon E(M/t(M)) \to M_t$ such that $h\nu\pi = \eta_M$. Conversely if $h\colon E(M/t(M)) \to M_t$ is an isomorphism and $h\nu\pi = \eta_M$ holds, then $E(M/t(M))/(M/t(M)) \simeq M_t/\eta_M(M) \in T(t)$ and so (*) is valid. Thus we have

Proposition 5.3 (cf. [1, Théorème 2]). For a left exact radical t the following conditions are equivalent:

(1) For each R-module M there exists an isomorphism h such

that $h\nu\pi = \eta_M$.

(2) For each R-module M in F(t), there exists an isomorphism h such that the diagram

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\eta_M"'] & E(M) \arrow[dl, "h"] \\ M_t & \end{tikzcd}$$

is commutative.

(3) If M is an R-module in F(t) and M' is an essential submodule, then M/M' is in T(t).

(4) If M is an R-module in F(t), then E(M)/M is in T(t).

It is to be noted that, if t = G, then the condition (3) (or (4)) is automatically valid and so we can claim that

$$M_G \simeq E(M/G(M))$$

for every R-module M.

The equivalence of (1) and (3) in the next proposition has given by [1, Théorème 3].

Proposition 5.4. For a left exact radical t the following conditions are equivalent:

(1) (i) T(t) is stable. (ii) If M is an R-module in F(t) and M'is an essential submodule, then M/M' is in T(t).

(2) (i) T(t) is closed under injective hulls. (ii) If M is an R-module in F(t), then E(M)/M is in T(t).

(3) $G \leqq t$.

(4) If M is an R-module and M' is an essential submodule, then M/M' is in T(t).

(5) If M is an R-module, then E(M)/M is in T(t).

(6) If $\underline{a}$ is an essential left ideal in R, then $R/\underline{a}$ is in T(t), i.e. $Z \leqq t$.

(7) If $\underline{a}$ is a left ideal in R, then $E(\underline{a})/\underline{a}$ is in $T(t)$.

Proof. We only show that (1) $\Rightarrow$ (4). Assume (1) and let M' be an essential submodule of M. First note that $t(M)$ is closed in M, i.e. $t(M) = E(t(M)) \cap M$. Since $E(t(M)) \in T(t)$, $E(t(M)) \cap M \in T(t)$. Hence $E(t(M)) \cap M \leqq t(M)$ and thus we have $t(M) = E(t(M)) \cap M$. Now $M' + t(M)$ is essential in M and so $(M' + t(M))/t(M)$ is also essential in $M/t(M)$ because of the closedness of $t(M)$. Hence by assumption $M/(M' + t(M)) \in T(t)$ and thus $M/M' \in T(t)$.

In addition to Theorem 3.2, the following theorem will also give some conditions that β_0 to be coincide with G.

Theorem 5.5. The following conditions are equivalent:

(1) $\beta_0 = G$.

(2) If M' is an essential submodule of an R-module M, then M/M' has no faithful prime images.

(3) For every R-module M, $E(M)/M$ has no faithful prime images.

(4) If $\underline{a}$ is an essential left ideal in R, then $R/\underline{a}$ has no faithful prime images.

(5) For every left ideal $\underline{a}$ in R, $E(\underline{a})/\underline{a}$ has no faithful prime images.

Finally in closing this note we shall give a relationship between two localizations R_{β_0} and R_β of a strongly prime ring R. Assume that R is strongly prime and put $R_{\beta_0} = Q_0$ and $R_G = Q_{max}(R) = Q$. Let $\eta: R \to Q$ and $\eta_0: R \to Q_0$ be the canonical maps. Then by Lemma 5.1 there exists an injective ring homomorphism $h: Q_0 \to Q$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\eta_0"] \arrow[d, "\eta"'] & Q_0 \arrow[dl, "h"] \\ Q & \end{tikzcd}$$

is commutative.

By [2, Proposition IV.1], Q_0 is also a strongly prime ring. Applying Lemma 5.1 to Q_0-mod, there also exists an injective ring homomorphism $k: Q \to Q_{max}(Q_0)$ making the diagram

$$\begin{tikzcd} Q_0 \arrow[r, "h"] \arrow[d, "\eta_m"'] & Q \arrow[dl, "k"] \\ Q_{max}(Q_0) & \end{tikzcd}$$

commutative, where η_m denotes the canonical map. Hence we have a sequence of ring extensions

$$R \leqq Q_0 \leqq Q \leqq Q_{max}(Q_0)$$

and $Q_{max}(Q_0)$ is a rational extension of Q. However Q has no proper rational extensions. Hence Q must be equal to $Q_{max}(Q_0)$. Thus we have

Theorem 5.6. If R is strongly prime, then $R_\beta = Q_{max}(R_{\beta_0})$.

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